Non-existence of static, spherically symmetric and stationary, axisymmetric traversable wormholes coupled to nonlinear electrodynamics

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In this work we explore the possible existence of static, spherically symmetric and stationary, axisymmetric traversable wormholes coupled to nonlinear electrodynamics. Considering static and spherically symmetric $(2+1)$ and $(3+1)-$dimensional wormhole spacetimes, we verify the presence of an event horizon and the non-violation of the null energy condition at the throat. For the former spacetime, the principle of finiteness is imposed, in order to obtain regular physical fields at the throat. Next, we analyze the $(2+1)-$dimensional stationary and axisymmetric wormhole, and also verify the presence of an event horizon, rendering the geometry non-traversable. Relatively to the $(3+1)-$dimensional stationary and axisymmetric wormhole geometry, we find that the field equations impose specific conditions that are incompatible with the properties of wormholes. Thus, we prove the non-existence of the general class of traversable wormhole solutions, outlined above, within the context of nonlinear electrodynamics.

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1. INTRODUCTION

A specific model of nonlinear electrodynamics was proposed by Born and Infeld in 1934 [1] founded on a principle of finiteness, namely, that a satisfactory theory should avoid physical quantities becoming infinite. The Born-Infeld model was inspired mainly to remedy the fact that the standard picture of a point particle possesses an infinite self-energy, and consisted on placing an upper limit on the electric field strength and considering a finite electron radius. Later, Plebański presented other examples of nonlinear electrodynamic Lagrangians [2], and showed that the Born-Infeld theory satisfies physically acceptable requirements. A further discussion of these properties can be found in Ref. [3]. Furthermore, a recent revival of nonlinear electrodynamics has been verified, mainly due to the fact that these theories appear as effective theories at different levels of string/M-theory, in particular, in D$p-$branes and supersymmetric extensions, and non-Abelian generalizations (see Ref. [4] for a review).

Much interest in nonlinear electrodynamic theories has also been aroused in applications to cosmological models, in particular, in explaining the inflationary epoch and the late accelerated expansion of the universe [5, 6]. In this cosmological context, an inhomogeneous and anisotropic nonsingular model for the universe, with a Born-Infeld field was studied [7], the effects produced by nonlinear electrodynamics in spacetimes conformal to Bianchi metrics were further analyzed [8], and geodesically complete Bianchi spaces were also found [9]. Homogeneous and isotropic cosmological solutions governed by the non-abelian Born-Infeld Lagrangian [10], and anisotropic cosmological spacetimes, in the presence of a positive cosmological constant [11], were also extensively analyzed. In fact, it is interesting to note that the first *exact* regular black hole solution in general relativity was found within nonlinear electrodynamics [12, 13], where the source is a nonlinear electrodynamic field satisfying the weak energy condition, and the Maxwell field is reproduced in the weak limit. It was also shown that general relativity coupled to nonlinear electrodynamics leads to regular magnetic black holes and monopoles [14], and regular electrically charged structures, possessing a regular de Sitter center [15], and the respective stability of these solutions was further explored in Ref. [16].

Recently, an alternative model to black holes was proposed, in particular, the gravastar picture [17], where there is an effective phase transition at or near where the event horizon is expected to form, and the interior is replaced by a de Sitter condensate. The gravastar model has no singularity at the origin and no event horizon, as its rigid surface is located at a radius slightly greater than the Schwarzschild radius. In this context, a gravastar model within nonlinear electrodynamics, where the interior de Sitter solution is substituted with a Born-Infeld Lagrangian, was also found. This solution was denoted as a Born-Infeld phantom gravastar [18].

Relatively to wormhole spacetimes [19, 20], an important and intriguing challenge is the quest to find a real-

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istic matter source that will support these exotic geometries. The latter are supported by *exotic matter*, involving a stress energy tensor that violates the null energy condition (NEC), i.e., $T_{\mu\nu}k^\mu k^\nu \geq 0$, where $T_{\mu\nu}$ is the stress-energy tensor and k^μ any null vector. Several candidates have been proposed in the literature, for instance, to cite a few, null energy condition violating massless conformally coupled scalar fields supporting self-consistent classical wormholes [21]; the extension of the Morris-Thorne wormhole with the inclusion of a cosmological constant [22]; and more recently, the theoretical realization that wormholes may be supported by exotic cosmic fluids, responsible for the accelerated expansion of the universe, such as phantom energy [23] and the generalized Chaplygin gas [24]. It is also interesting to note that an effective wormhole geometry for an electromagnetic wave can appear as a result of the nonlinear character of the field [25].

In Ref [26], evolving $(2+1)$ and $(3+1)-$dimensional wormhole spacetimes, conformally related to the respective static geometries, within the context of nonlinear electrodynamics were also explored. It was found that for the specific $(3+1)-$dimensional spacetime, the Einstein field equation imposes a contracting wormhole solution and the obedience of the weak energy condition. Furthermore, in the presence of an electric field, the latter presents a singularity at the throat. However, a regular solution was found for a pure magnetic field. For the $(2+1)-$dimensional case, it was also found that the physical fields are singular at the throat. Thus, taking into account the principle of finiteness, that a satisfactory theory should avoid physical quantities becoming infinite, one may rule out evolving $(3+1)-$dimensional wormhole solutions, in the presence of an electric field, and the $(2+1)-$dimensional case coupled to nonlinear electrodynamics.

In this work we shall be interested in exploring the possibility that nonlinear electrodynamics may support static, spherically symmetric and stationary, axisymmetric traversable wormhole geometries. In fact, Bronnikov [14, 27] showed that static and spherically symmetric $(3+1)-$dimensional wormholes is not the case, and we shall briefly reproduce and confirm this result. We further consider the $(2+1)-$dimensional case, which proves to be extremely interesting, as the principle of finiteness is imposed, in order to obtain regular physical fields at the throat. We shall next analyze the $(2+1)$ and $(3+1)-$dimensional stationary and axisymmetric case [28] coupled to nonlinear electrodynamics.

This paper is outlined in the following manner: In Sec. 2 we analyze $(2+1)$ and $(3+1)-$dimensional static and spherically symmetric wormholes coupled with nonlinear electrodynamics, and in Sec. 3 rotating traversable wormholes in the context of nonlinear electrodynamics are studied. In Sec. 4 we conclude.

2. STATIC AND SPHERICALLY SYMMETRIC WORMHOLES

2.1. $(2+1)-$dimensional wormhole

In this Section, we shall be interested in $(2+1)-$dimensional general relativity coupled to nonlinear electrodynamics. We will use geometrized units throughout this work, i.e., $G=c=1$. The respective action is given by

$$S=\int \sqrt{-g}\left[\frac{R}{16\pi}+L(F)\right]\,d^3x\,, \tag{1}$$

where R is the Ricci scalar and $L(F)$ is a gauge-invariant electromagnetic Lagrangian, which we shall leave unspecified at this stage, depending on the invariant F given by $F=\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$ is the electromagnetic field. Note that the factor $1/16\pi$, in the action, is maintained to keep the parallelism with $(3+1)-$dimensional theory [29].

In Einstein-Maxwell theory, the Lagrangian is defined as $L(F)\equiv -F/4\pi$, but here we consider more general choices of electromagnetic Lagrangians, however, depending on the single invariant F. It is perhaps important to emphasize that we do not consider the case where L depends on the invariant $G\equiv \frac{1}{4}F_{\mu\nu}{}^{*}F^{\mu\nu}$, where $*$ denotes the Hodge dual with respect to $g_{\mu\nu}$.

Varying the action with respect to the gravitational field provides the Einstein tensor

$$G_{\mu\nu}=8\pi(g_{\mu\nu}L-F_{\mu\alpha}F_{\nu}{}^{\alpha}\,L_F)\,, \tag{2}$$

where $L_F\equiv dL/dF$. Clearly, the stress-energy tensor is given by

$$T_{\mu\nu}=g_{\mu\nu}\,L(F)-F_{\mu\alpha}F_{\nu}{}^{\alpha}\,L_F\,, \tag{3}$$

where the Einstein field equation is defined as $G_{\mu\nu}=8\pi T_{\mu\nu}$. The variation of the action with respect to the electromagnetic potential A_μ, yields the electromagnetic field equations

$$\left(F^{\mu\nu}\,L_F\right)_{;\mu}\;=\;0\,, \tag{4}$$

where the semi-colon denotes a covariant derivative.

The spacetime metric representing a spherically symmetric and static $(2+1)-$dimensional wormhole is given by

$$ds^2=-e^{2\Phi(r)}\,dt^2+\frac{dr^2}{1-b(r)/r}+r^2\,d\phi^2\,, \tag{5}$$

where $\Phi(r)$ and $b(r)$ are functions of the radial coordinate, r. $\Phi(r)$ is denoted as the redshift function, for it is related to the gravitational redshift; $b(r)$ is called the form function [19]. The radial coordinate has a range that increases from a minimum value at r_0, corresponding to the wormhole throat, to ∞.

For the wormhole to be traversable, one must demand the absence of event horizons, which are identified as the surfaces with $e^{2\Phi} \to 0$, so that $\Phi(r)$ must be finite everywhere. A fundamental property of wormhole physics is the flaring out condition, which is deduced from the mathematics of embedding, and is given by $(b - b'r)/b^2 > 0$ [19, 30]. Note that at the throat $b(r_0) = r = r_0$, the flaring out condition reduces to $b'(r_0) < 1$. The condition $(1 - b/r) > 0$ is also imposed.

Taking into account the symmetries of the geometry, we shall consider the following electromagnetic tensor

$$F_{\mu\nu} = E(r)(\delta^t_\mu \delta^r_\nu - \delta^r_\mu \delta^t_\nu) + B(r)(\delta^\phi_\mu \delta^r_\nu - \delta^r_\mu \delta^\phi_\nu) \,. \qquad (6)$$

Note that the only non-zero terms for the electromagnetic tensor are the following $F_{tr} = -F_{rt} = E(r)$ and $F_{\phi r} = -F_{r\phi} = B(r)$. The invariant $F = F^{\mu\nu}F_{\mu\nu}/4$ is given by

$$F = -\frac{1}{2}\left(1 - \frac{b}{r}\right)\left[e^{-2\Phi}\, E^2(r) - \frac{1}{r^2}\, B^2(r)\right] . \qquad (7)$$

The electromagnetic field equation, Eq. (4), provides the following relationships

$$e^{-\Phi}\left(1 - \frac{b}{r}\right)^{1/2} E\, L_F \;=\; \frac{C_e}{r}\,, \qquad (8)$$

$$\frac{1}{r}\left(1 - \frac{b}{r}\right)^{1/2} B\, L_F \;=\; C_m\, e^{-\Phi}\,. \qquad (9)$$

where the constants of integration C_e and C_m are related to the electric and magnetic charge, q_e and q_m, respectively.

The mathematical analysis and the physical interpretation will be simplified using a set of orthonormal basis vectors. These may be interpreted as the proper reference frame of a set of observers who remain at rest in the coordinate system (t, r, ϕ), with (r, ϕ) fixed. Now, the non-zero components of the Einstein tensor, $G_{\hat{\mu}\hat{\nu}}$, in the orthonormal reference frame, are given by

$$G_{\hat{t}\hat{t}} \;=\; \frac{b'r - b}{2r^3}\,, \qquad (10)$$

$$G_{\hat{r}\hat{r}} \;=\; \left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\,, \qquad (11)$$

$$G_{\hat{\phi}\hat{\phi}} \;=\; \left(1 - \frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)}\Phi'\right] . \qquad (12)$$

The Einstein field equation, $G_{\hat{\mu}\hat{\nu}} = 8\pi\, T_{\hat{\mu}\hat{\nu}}$, requires that the Einstein tensor be proportional to the stress-energy tensor, so that in the orthonormal basis the latter must have an identical algebraic structure as the Einstein tensor components, $G_{\hat{\mu}\hat{\nu}}$, i.e., Eqs. (10)-(12).

Recall that a fundamental condition in wormhole physics is the violation of the NEC, which is defined as $T_{\mu\nu}k^\mu k^\nu \geq 0$, where k^μ is *any* null vector. Considering the orthonormal reference frame with $k^{\hat{\mu}} = (1, \pm 1, 0)$, we have

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \frac{1}{8\pi}\left[\frac{b'r - b}{r^3} + \left(1 - \frac{b}{r}\right)\frac{\Phi'}{r}\right] . \qquad (13)$$

Using the flaring out condition of the throat, $(b - b'r)/2b^2 > 0$ [19, 20], and considering the finite character of $\Phi(r)$, we verify that evaluated at the throat the NEC is violated, i.e., $T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} < 0$. Matter that violates the NEC is denoted as *exotic matter*.

The only non-zero components of $T_{\hat{\mu}\hat{\nu}}$, taking into account Eq. (3), are

$$T_{\hat{t}\hat{t}} \;=\; -L - e^{-2\Phi}\left(1 - \frac{b}{r}\right) E^2\, L_F\,, \qquad (14)$$

$$T_{\hat{r}\hat{r}} \;=\; L + e^{-2\Phi}\left(1 - \frac{b}{r}\right) E^2\, L_F - \left(1 - \frac{b}{r}\right)\frac{B^2}{r^2}\, L_F\,, \qquad (15)$$

$$T_{\hat{\phi}\hat{\phi}} \;=\; L - \left(1 - \frac{b}{r}\right)\frac{B^2}{r^2}\, L_F\,. \qquad (16)$$

We need to impose the conditions $|e^{-2\phi}(1 - b/r)E^2 L_F| < \infty$ and $|(1 - b/r)B^2 L_F| < \infty$ as $r \to r_0$, to ensure the regularity of the stress-energy tensor components.

Note that the Lagrangian may be obtained from the following relationship: $L = T_{\hat{\phi}\hat{\phi}} - T_{\hat{t}\hat{t}} - T_{\hat{r}\hat{r}}$, and using the Einstein field equation, is given by

$$L \;=\; \frac{1}{8\pi}\left\{\left(1 - \frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{\Phi'}{r} - \frac{b'r - b}{2r(r-b)}\Phi'\right] - \frac{b'r - b}{2r^3}\right\} . \qquad (17)$$

However, from the metric (5) we verify the following zero components of the Einstein tensor: $G_{\hat{t}\hat{r}} = 0$, $G_{\hat{r}\hat{\phi}} = 0$ and $G_{\hat{t}\hat{\phi}} = 0$. Thus, through the Einstein field equation, a further restriction may be obtained from $T_{\hat{t}\hat{\phi}} = 0$, i.e.,

$$T_{\hat{t}\hat{\phi}} = -\frac{1}{r}E(r)B(r)e^{-\phi}(1 - b/r)\, L_F\,, \qquad (18)$$

which imposes that $E(r) = 0$ or $B(r) = 0$, considering the non-trivial case of L_F non-zero. It is rather interesting that both $E(r)$ and $B(r)$ cannot coexist simultaneously, in the present $(2+1)-$dimensional case.

For the specific case of $B(r) = 0$, from Eqs. (10)-(11) and Eqs. (14)-(15), we verify the following condition

$$\Phi' = -\frac{b'r - b}{2r(r-b)}\,, \qquad (19)$$

which may be integrated to yield the solution

$$e^{2\Phi} = \left(1 - \frac{b}{r}\right) . \qquad (20)$$

This corresponds to a non-traversable wormhole solution, as it possesses an event horizon at the throat, $r = r_0$.

Now, consider the case of $E(r) = 0$ and $B(r) \neq 0$. For this case we have $T_{\hat{r}\hat{r}} = T_{\hat{\phi}\hat{\phi}}$, and the respective Einstein tensor components, Eqs. (11)-(12), provide the following differential equation

$$\frac{\Phi'}{r} = \Phi'' + (\Phi')^2 - \frac{b'r - b}{2r(r-b)}\Phi' \,. \qquad (21)$$

Considering a specific choice of $b(r)$ or $\Phi(r)$, one may, in principle, obtain a solution for the geometry. Equation (21) may be formally integrated to yield the following general solution

$$\Phi(r) = \ln\left[C_1 \int r\left(1 - \frac{b(r)}{r}\right)^{-1/2} dr + C_2\right] \,, \qquad (22)$$

where C_1 and C_2 are constants of integration. For instance, consider a constant form function, $b(r) = r_0$, so that from Eq. (22), we deduce

$$\begin{aligned}\Phi(r) \;&=\; \ln\Big\{C_2 + \frac{C_1}{8}\Big[2\sqrt{r(r-r_0)}\,(2r+3r_0) \\ &\quad +3r_0^2\ln\left(r - r_0/2 + \sqrt{r(r-r_0)}\right)\Big]\Big\} \,, \qquad (23)\end{aligned}$$

which at the throat reduces to

$$\Phi(r_0) = \ln\left[C_2 + \frac{3C_1 r_0^2}{8}\ln\left(\frac{r_0}{2}\right)\right] \,. \qquad (24)$$

To obtain a regular solution at the throat, we impose the condition: $C_2 + (3C_1 r_0^2/8)\ln(r_0/2) > 0$.

Consider for instance $b(r) = r_0^2/r$, then Eq. (22) provides the solution

$$\begin{aligned}\Phi(r) \;&=\; \ln\left\{C_2 + \frac{C_1}{2}\left[r\sqrt{r^2 - r_0^2}\right.\right. \\ &\quad \left.\left. +r_0^2\ln\left(r + \sqrt{r^2 - r_0^2)}\right)\right]\right\} \,, \qquad (25)\end{aligned}$$

which at the throat, reduces to

$$\Phi(r_0) = \ln\left[C_2 + \frac{C_1 r_0^2}{2}\ln(r_0)\right] \,. \qquad (26)$$

Once again, to ensure a regular solution, we need to impose the following condition: $C_2 + (C_1 r_0^2/2)\ln(r_0) > 0$. Note that these specific solutions are not asymptotically flat, however, they may be matched to an exterior vacuum spacetime, much in the spirit of Refs. [22, 31].

However, a subtlety needs to be pointed out. Consider Eqs. (10)-(11) and (14)-(15), from which we deduce

$$\Phi' = -\frac{b'r - b}{2r(r-b)} - \frac{8\pi B^2}{r}\,L_F \,. \qquad (27)$$

Now, taking into account Eq. (9), we find the following relationships for the magnetic field, $B(r)$, and for L_F

$$B(r) = -\frac{e^{\Phi}}{8\pi C_m}\left[\frac{b'r - b}{2r^2(1-b/r)^{1/2}} + \left(1 - \frac{b}{r}\right)^{1/2}\Phi'\right] \,, \qquad (28)$$

and

$$L_F = -\frac{8\pi C_m^2 r\, e^{-2\Phi}}{\frac{b'r-b}{2r^2} + (1 - b/r)\,\Phi'} \,, \qquad (29)$$

respectively. Considering that the redshift Φ be finite throughout the spacetime, one immediately verifies that the magnetic field $B(r)$ is singular at the throat, which is transparent considering the first term in square brackets in the right hand side of Eq. (28). This is an extremely troublesome aspect of the geometry, as in order to construct a traversable wormhole, singularities appear in the physical fields. This aspect is in contradiction to the model construction of nonlinear electrodynamics, founded on a principle of finiteness, that a satisfactory theory should avoid physical quantities becoming infinite [1]. Thus, one should impose that these physical quantities be non-singular, and in doing so, we verify that the general solution corresponds to a non-traversable wormhole geometry. This may be verified by integrating Eq. (27), which yields the following general solution

$$e^{2\Phi} = \left(1 - \frac{b}{r}\right)\exp\left(-16\pi\int \frac{B^2}{r}\,L_F\,dr\right) \,. \qquad (30)$$

We have considered the factor $|B^2 L_F| < \infty$ as $r \to r_0$, to ensure the regularity of the term in the exponential. However, this solution corresponds to a non-traversable wormhole solution, as it possesses an horizon at the throat, $b = r = r_0$.

One may also prove the non-existence of $(2+1)-$dimensional static and spherically symmetric traversable wormholes in nonlinear electrodynamics, through an analysis of the NEC violation. In the context of nonlinear electrodynamics, and taking into account Eqs. (14)-(15), we verify

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = -\left(1 - \frac{b}{r}\right)\frac{B^2}{r^2}\,L_F \,, \qquad (31)$$

which evaluated at the throat, considering the regularity of B and L_F, is identically zero, i.e., $T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}}|_{r_0} = 0$. The NEC is not violated at the throat, so that the flaring-out condition is not satisfied, showing, therefore, the non-existence of $(2+1)-$dimensional static and spherically symmetric traversable wormholes in nonlinear electrodynamics.

2.2. $(3+1)-$dimensional wormhole

The action of $(3+1)-$dimensional general relativity coupled to nonlinear electrodynamics is given by

$$S = \int \sqrt{-g}\left[\frac{R}{16\pi} + L(F)\right]\,d^4x \,, \qquad (32)$$

where R is the Ricci scalar and the gauge-invariant electromagnetic Lagrangian, $L(F)$, depends on a single invariant F [2, 32], defined by $F \equiv \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, as before. We shall not consider the case where L depends on the invariant $G \equiv \frac{1}{4}F_{\mu\nu}{}^{*}F^{\mu\nu}$, as mentioned in the $(2+1)-$dimensional case.

Varying the action with respect to the gravitational field provides the Einstein field equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, where the stress-energy tensor is given by

$$T_{\mu\nu} = g_{\mu\nu}\, L(F) - F_{\mu\alpha}F_{\nu}{}^{\alpha}\, L_F \,. \tag{33}$$

Taking into account the symmetries of the geometry, the only non-zero compatible terms for the electromagnetic tensor are $F_{tr} = E(x^\mu)$ and $F_{\theta\phi} = B(x^\mu)$.

The spacetime metric representing a spherically symmetric and static $(3+1)-$dimensional wormhole takes the form [19]

$$ds^2 = -e^{2\Phi(r)}\, dt^2 + \frac{dr^2}{1-b(r)/r} + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{34}$$

The non-zero components of the Einstein tensor, given in an orthonormal reference frame, are given by

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2}\,, \tag{35}$$

$$G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + 2\left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\,, \tag{36}$$

$$G_{\hat{\phi}\hat{\phi}} = G_{\hat{\theta}\hat{\theta}} = \left(1-\frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r-b}{2r(r-b)}\Phi' - \frac{b'r-b}{2r^2(r-b)} + \frac{\Phi'}{r}\right]. \tag{37}$$

Its a simple matter to prove that for this geometry, the NEC is identical to Eq. (13), and is also violated at the throat, i.e., $T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} < 0$.

The relevant components for the stress-energy tensor, regarding the analysis of the NEC, are the following

$$T_{\hat{t}\hat{t}} = -L - e^{-2\Phi}\left(1-\frac{b}{r}\right)E^2\, L_F\,, \tag{38}$$

$$T_{\hat{r}\hat{r}} = L + e^{-2\Phi}\left(1-\frac{b}{r}\right)E^2\, L_F\,. \tag{39}$$

Analogously with the $(2+1)-$dimensional case, we will consider $|e^{-2\Phi}(1-b/r)\,E^2\,L_F| < \infty$, as $r \to r_0$, to ensure that the stress-energy tensor components are regular.

From Eqs. (35)-(36) and Eqs. (38)-(39), we verify the following condition

$$\Phi' = -\frac{b'r-b}{2r(r-b)}\,, \tag{40}$$

which may be integrated to yield the solution $e^{2\Phi} = (1-b/r)$, rendering a non-traversable wormhole solution, as it possesses an event horizon at the throat, $r = r_0$.

Note that the NEC, for the stress-energy tensor defined by (33), is identically zero for arbitrary r, i.e., $T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = 0$. In particular this implies that the flaring-out condition of the throat is not satisfied, showing, therefore, the non-existence of $(3+1)-$dimensional static and spherically symmetric traversable wormholes coupled to nonlinear electrodynamics. The analysis outlined in this Section is consistent with that of Refs. [14, 27], where it was pointed out that nonlinear electrodynamics, with any Lagrangian of the form $L(F)$, coupled to general relativity cannot support static and spherically symmetric $(3+1)-$dimensional traversable wormholes.

The impediment to the construction of traversable wormholes may be overcome by considering a non-interacting anisotropic distribution of matter coupled to nonlinear electrodynamics. This may be reflected by the following superposition of the stress-energy tensor

$$T_{\mu\nu} = T_{\mu\nu}^{\rm fluid} + T_{\mu\nu}^{\rm NED}\,, \tag{41}$$

where $T_{\mu\nu}^{\rm NED}$ is given by Eq. (33), and $T_{\mu\nu}^{\rm fluid}$ is provided by

$$T_{\mu\nu}^{\rm fluid} = (\rho + p_t)U_\mu\, U_\nu + p_t\, g_{\mu\nu} + (p_r - p_t)\chi_\mu\chi_\nu\,. \tag{42}$$

U^μ is the four-velocity and χ^μ is the unit spacelike vector in the radial direction. $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure measured in the direction of χ^μ, and $p_t(r)$ is the transverse pressure measured in the orthogonal direction to χ^μ.

Now, the NEC takes the form

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \rho(r) + p_r(r) = \frac{1}{8\pi}\left[\frac{b'r-b}{r^3} + \left(1-\frac{b}{r}\right)\frac{\Phi'}{r}\right], \tag{43}$$

which evaluated at the throat, reduces to the NEC violation analysis of Ref. [19], i.e., $\rho + p_r < 0$.

3. STATIONARY AND AXISYMMETRIC WORMHOLES

3.1. $(2+1)-$dimensional wormhole

We now analyze nonlinear electrodynamics coupled to a stationary axisymmetric $(2+1)-$dimensional wormhole geometry. The stationary character of the spacetime implies the presence of a time-like Killing vector field, generating invariant time translations. The axially symmetric character of the geometry implies the existence of a spacelike Killing vector field, generating invariant rotations with respect to the angular coordinate ϕ. Consider the metric

$$ds^2 = -N^2dt^2 + \frac{dr^2}{1-b/r} + r^2K^2(d\phi - \omega\, dt)^2\,, \tag{44}$$

where N, K, ω and b are functions of r. $\omega(r)$ may be interpreted as the angular velocity $d\phi/dt$ of a particle. N is the analog of the redshift function in Eq. (5) and is finite and nonzero to ensure that there are no event horizons. We shall also assume that $K(r)$ is a positive, nondecreasing function of r that determines the proper radial distance R, i.e.,

$$R \equiv rK\,, \qquad R' > 0\,. \tag{45}$$

To transform to an orthonormal reference frame, the one-forms in the orthonormal basis transform as $\tilde{\Theta}^{\hat{\mu}} = \Lambda^{\hat{\mu}}{}_{\nu}\,\Theta^{\nu}$. The metric (44) can be diagonalized

$$ds^2 = -(\Theta^{\hat{t}})^2 + (\Theta^{\hat{r}})^2 + (\Theta^{\hat{\phi}})^2\,, \tag{46}$$

by means of the tetrad

$$\Theta^{\hat{t}} = N dt\,, \tag{47}$$

$$\Theta^{\hat{r}} = (1-b/r)^{-1/2} dr\,, \tag{48}$$

$$\Theta^{\hat{\phi}} = rK(d\phi - \omega dt)\,. \tag{49}$$

Now, $\Lambda^{\mu}{}_{\hat{\alpha}}\,\Lambda^{\hat{\alpha}}{}_{\nu} = \delta^{\mu}{}_{\nu}$ and $\Lambda^{\mu}{}_{\hat{\nu}}$ is defined as

$$(\Lambda^{\mu}{}_{\hat{\nu}}) = \begin{bmatrix} 1/N & 0 & 0 \\ 0 & (1-b/r)^{1/2} & 0 \\ \omega/N & 0 & (rK)^{-1} \end{bmatrix}\,. \tag{50}$$

From the latter transformation, one may deduce the orthonormal basis vectors, $\mathbf{e}_{\hat{\mu}} = \Lambda^{\nu}{}_{\hat{\mu}}\,\mathbf{e}_{\nu}$, given by

$$\mathbf{e}_{\hat{t}} = \frac{1}{N}\,\mathbf{e}_t + \frac{\omega}{N}\,\mathbf{e}_{\phi}\,, \tag{51}$$

$$\mathbf{e}_{\hat{r}} = \left(1 - \frac{b}{r}\right)^{1/2}\mathbf{e}_r\,, \tag{52}$$

$$\mathbf{e}_{\hat{\phi}} = \frac{1}{rK}\,\mathbf{e}_{\phi}\,. \tag{53}$$

Using the fact that $\mathbf{e}_{\alpha}\cdot\mathbf{e}_{\beta} = g_{\alpha\beta}$, we have $\mathbf{e}_{\hat{\mu}}\cdot\mathbf{e}_{\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$.

Using the Einstein field equation, $G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}$ and taking into account the null vector $k^{\hat{\mu}} = (1, \pm 1, 0)$, we obtain the following relationship, at the throat

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}}\,k^{\hat{\nu}} = -\frac{1-b'}{16\pi r_0^2 K}(K + r_0 K')\,. \tag{54}$$

We verify that the NEC is clearly violated because the conditions $K > 0$ and $K' > 0$ are imposed by construction [28], so that the metric (44) can describe a wormhole type solution.

Now for nonlinear electrodynamics, we consider the stress energy tensor given by Eq. (3), where the nonzero components of the electromagnetic tensor are

$$F_{tr} = -F_{rt}, \quad F_{t\phi} = -F_{\phi t}, \quad F_{\phi r} = -F_{r\phi}\,, \tag{55}$$

which are only functions of the radial coordinate r. Then, using the orthonormal reference frame, the NEC takes the following form

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}}\,k^{\hat{\nu}} = -\left[N^2\left(1 - \frac{b}{r}\right)F_{tr}^2 + F_{t\phi}^2\right]\frac{L_F}{r^2K^2N^2}\,, \tag{56}$$

which, at the throat, reduces to

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}}\,k^{\hat{\nu}}\big|_{r_0} = -\frac{1}{r_0^2 K^2 N^2}F_{t\phi}^2\,L_F\,. \tag{57}$$

From this relationship, we verify that the NEC is violated at the throat only if the derivative L_F is positive and $F_{t\phi}$ is nonzero. The latter condition, $F_{t\phi} \neq 0$, is imposed to have a compatibility of Eqs. (54) and (57).

However, note that from the metric (44) we verify the following zero components of the Einstein tensor: $G_{tr} = 0$ and $G_{r\phi} = 0$, implying that $T_{tr} = 0$ and $T_{r\phi} = 0$. These stress-energy tensor components are given by

$$T_{tr} = -(F_{tr}\,g^{t\phi} + F_{r\phi}\,g^{\phi\phi})\,F_{t\phi}\,L_F\,, \tag{58}$$

$$T_{r\phi} = -(F_{tr}\,g^{tt} + F_{r\phi}\,g^{t\phi})\,F_{t\phi}\,L_F\,. \tag{59}$$

From these conditions, considering that the derivative L_F be finite and positive and the non-trivial case $F_{t\phi} \neq 0$, we find that

$$(g^{t\phi})^2 = g^{tt}g^{\phi\phi}\,. \tag{60}$$

From the above imposition we deduce $N = 0$, implying the presence of an event horizon, showing the nonexistence of $(2+1)-$dimensional stationary and axially symmetric traversable wormholes coupled to nonlinear electrodynamics.

3.2. $(3+1)-$dimensional wormhole

Now, consider the stationary and axially symmetric $(3+1)-$dimensional spacetime, and analogously to the previous case, it possesses a time-like Killing vector field, which generates invariant time translations, and a space-like Killing vector field, which generates invariant rotations with respect to the angular coordinate ϕ. We have the following metric

$$ds^2 = -N^2 dt^2 + e^{\mu}\,dr^2 + r^2K^2[d\theta^2 + \sin^2\theta(d\phi - \omega\,dt)^2] \tag{61}$$

where N, K, ω and μ are functions of r and θ [28]. $\omega(r,\theta)$ may be interpreted as the angular velocity $d\phi/dt$ of a particle that falls freely from infinity to the point (r,θ). For simplicity, we shall consider the definition [28]

$$e^{-\mu(r,\theta)} = 1 - \frac{b(r,\theta)}{r}\,, \tag{62}$$

which is well suited to describe a traversable wormhole. Assume that $K(r,\theta)$ is a positive, nondecreasing function of r that determines the proper radial distance R, i.e., $R \equiv rK$ and $R_r > 0$ [28], as for the $(2+1)-$dimensional case. We shall adopt the notation that the subscripts $_r$ and $_\theta$ denote the derivatives in order of r and θ, respectively [28].

We shall also write down the contravariant metric tensors, which will be used later, and are given by

$$g^{tt} = -\frac{1}{N^2}\,, \quad g^{rr} = \left(1-\frac{b}{r}\right)\,, \quad g^{\theta\theta} = \frac{1}{r^2K^2}\,,$$
$$g^{\phi\phi} = \frac{N^2 - r^2\omega^2K^2\sin^2\theta}{r^2N^2K^2\sin^2\theta}\,, \quad g^{t\phi} = -\frac{\omega}{N^2}\,. \tag{63}$$

Note that an event horizon appears whenever $N = 0$ [28]. The regularity of the functions N, b and K are imposed, which implies that their θ derivatives vanish on the rotation axis, $\theta = 0, \pi$, to ensure a non-singular behavior of the metric on the rotation axis. The metric (61) reduces to the Morris-Thorne spacetime metric (5) in the limit of zero rotation and spherical symmetry

$$N(r,\theta) \to \mathrm{e}^{\Phi(r)}, \quad b(r,\theta) \to b(r)\,, \tag{64}$$
$$K(r,\theta) \to 1\,, \quad \omega(r,\theta) \to 0\,. \tag{65}$$

In analogy with the Morris-Thorne case, $b(r_0) = r_0$ is identified as the wormhole throat, and the factors N, K and ω are assumed to be well-behaved at the throat.

The scalar curvature of the space-time (61) is extremely messy, but at the throat $r = r_0$ simplifies to

$$\begin{aligned} R \;=\; & -\frac{1}{r^2K^2}\left(\mu_{\theta\theta}+\frac{1}{2}\mu_\theta^2\right) - \frac{\mu_\theta}{Nr^2K^2}\,\frac{(N\sin\theta)_\theta}{\sin\theta} \\ & -\frac{2}{Nr^2K^2}\,\frac{(N_\theta\sin\theta)_\theta}{\sin\theta} - \frac{2}{r^2K^3}\,\frac{(K_\theta\sin\theta)_\theta}{\sin\theta} \\ & +e^{-\mu}\,\mu_r\,\left[\ln(Nr^2K^2)\right]_r + \frac{\sin^2\theta\,\omega_\theta^2}{2N^2} \\ & +\frac{2}{r^2K^4}\,(K^2+K_\theta^2)\,. \end{aligned} \tag{66}$$

The only troublesome terms are the ones involving the terms with μ_θ and $\mu_{\theta\theta}$, i.e.,

$$\mu_\theta = \frac{b_\theta}{(r-b)}\,, \qquad \mu_{\theta\theta}+\frac{1}{2}\mu_\theta^2 = \frac{b_{\theta\theta}}{r-b}+\frac{3}{2}\frac{{b_\theta}^2}{(r-b)^2}\,. \tag{67}$$

Note that one needs to impose that $b_\theta = 0$ and $b_{\theta\theta} = 0$ at the throat to avoid curvature singularities. This condition shows that the throat is located at a constant value of r.

Thus, one may conclude that the metric (61) describes a rotating wormhole geometry, with an angular velocity ω. The factor K determines the proper radial distance. N is the analog of the redshift function in Eq. (34) and is finite and nonzero to ensure that there are no event horizons or curvature singularities. b is the shape function which satisfies $b \leq r$; it is independent of θ at the throat, i.e., $b_\theta = 0$; and obeys the flaring out condition $b_r < 1$.

In the context of nonlinear electrodynamics, we consider the stress energy tensor defined in Eq. (33). The nonzero components of the electromagnetic tensor are

$$F_{tr} = -F_{rt}, F_{t\theta} = -F_{\theta t}, F_{t\phi} = -F_{\phi t}, \tag{68}$$
$$F_{\phi r} = -F_{r\phi}, F_{r\theta} = -F_{\theta r}, F_{\theta\phi} = -F_{\phi\theta} \tag{69}$$

which are functions of the radial coordinate r and the angular coordinate θ.

The analysis is simplified using an orthonormal reference frame, with the following orthonormal basis vectors

$$\mathbf{e}_{\hat{t}} \;=\; \frac{1}{N}\,\mathbf{e}_t + \frac{\omega}{N}\,\mathbf{e}_\phi\,, \tag{70}$$
$$\mathbf{e}_{\hat{r}} \;=\; \left(1-\frac{b}{r}\right)^{1/2}\,\mathbf{e}_r\,, \tag{71}$$
$$\mathbf{e}_{\hat{\theta}} \;=\; \frac{1}{rK}\,\mathbf{e}_\theta\,, \tag{72}$$
$$\mathbf{e}_{\hat{\phi}} \;=\; \frac{1}{rK\sin\theta}\,\mathbf{e}_\phi\,. \tag{73}$$

Now the Einstein tensor components are extremely messy, but assume a more simplified form using the orthonormal reference frame and evaluated at the throat. They have the following non-zero components

$$\begin{aligned} G_{\hat{t}\hat{t}} \;=\; & -\frac{(K_\theta\sin\theta)_\theta}{r^2K^3\sin\theta} - \frac{\omega_\theta^2\,\sin^2\theta}{4N^2} + e^{-\mu}\,\mu_r\,\frac{(rK)_r}{rK} \\ & +\frac{K^2+K_\theta^2}{r^2K^4}\,, \end{aligned} \tag{74}$$

$$\begin{aligned} G_{\hat{r}\hat{r}} \;=\; & \frac{(K_\theta\sin\theta)_\theta}{r^2K^3\sin\theta} - \frac{\omega_\theta^2\,\sin^2\theta}{4N^2} + \frac{(N_\theta\sin\theta)_\theta}{Nr^2K^2\sin\theta} \\ & -\frac{K^2+K_\theta^2}{r^2K^4}\,, \end{aligned} \tag{75}$$

$$G_{\hat{r}\hat{\theta}} \;=\; \frac{e^{-\mu/2}\,\mu_\theta\,(rKN)_r}{2Nr^2K^2}\,, \tag{76}$$

$$\begin{aligned} G_{\hat{\theta}\hat{\theta}} \;=\; & \frac{N_\theta(K\sin\theta)_\theta}{Nr^2K^3\sin\theta} + \frac{\omega_\theta^2\,\sin^2\theta}{4N^2} \\ & -\frac{\mu_r\,e^{-\mu}(NrK)_r}{2NrK}\,, \end{aligned} \tag{77}$$

$$\begin{aligned} G_{\hat{\phi}\hat{\phi}} \;=\; & -\frac{\mu_r\,e^{-\mu}\,(NKr)_r}{2NKr} - \frac{3\sin^2\theta\,\omega_\theta^2}{4N^2} \\ & +\frac{N_{\theta\theta}}{Nr^2K^2} - \frac{N_\theta K_\theta}{Nr^2K^3}\,, \end{aligned} \tag{78}$$

$$\begin{aligned} G_{\hat{t}\hat{\phi}} \;=\; & \frac{1}{4N^2K^2r}\,\Big(6NK\,\omega_\theta\,\cos\theta + 2NK\,\sin\theta\,\omega_{\theta\theta} \\ & -\mu_r e^{-\mu} r^2 NK^3\,\sin\theta\,\omega_r + 4N\,\omega_\theta\,\sin\theta\,K_\theta \\ & -2K\,\sin\theta\,N_\theta\,\omega_\theta\Big)\,. \end{aligned} \tag{79}$$

Note that the component $G_{\hat{r}\hat{\theta}}$ is zero at the throat, however, we have included this term, as it shall be helpful in the analysis of the stress-energy tensor components, outlined below.

Using the Einstein field equation, the components $T_{\hat{t}\hat{t}}$ and $T_{\hat{i}\hat{j}}$ have the usual physical interpretations, and in particular, $T_{\hat{t}\hat{\phi}}$ characterizes the rotation of the matter distribution. It is interesting to note that constraints on the geometry, placing restrictions on the stress energy tensor needed to generate a general stationary and axisymmetric spacetime, were found in Ref. [33]. Taking into account the Einstein tensor components above, the

NEC at the throat is given by

$$8\pi\, T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = \mathrm{e}^{-\mu}\mu_r \frac{(rK)_r}{rK} - \frac{{\omega_\theta}^2 \sin^2\theta}{2N^2} + \frac{(N_\theta \sin\theta)_\theta}{(rK)^2 N \sin\theta}\,. \tag{80}$$

Rather than reproduce the analysis here, we refer the reader to Ref. [28], where it was shown that the NEC is violated in certain regions, and is satisfied in others. Thus, it is possible for an infalling observer to move around the throat, and avoid the exotic matter supporting the wormhole. However, it is important to emphasize that one cannot avoid the use of exotic matter altogether.

Using the stress-energy tensor, Eq. (33), we verify the following relationship

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = -\Big[F_{t\phi}^2 + \sin^2\theta (F_{t\theta} + \omega F_{\phi\theta})^2 + \left(1-\frac{b}{r}\right) N^2 (F_{\phi r}^2 + \sin^2\theta F_{r\theta}^2)\Big] \frac{L_F}{r^2 K^2 N^2 \sin^2\theta}\,, \tag{81}$$

which evaluated at the throat reduces to

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = -[F_{t\phi}^2 + \sin^2\theta (F_{t\theta} + \omega F_{\phi\theta})^2] \frac{L_F}{r_0^2 K^2 N^2 \sin^2\theta}\,. \tag{82}$$

Note that for this expression to be compatible with Eq. (80), L_F may be either positive, negative or zero.

The non-zero components of the Einstein tensor are precisely the components expressed in Eqs. (74)-(79), so that through the Einstein field equation, we have the following zero components for the stress energy tensor in the (3+1)−dimensional case: $T_{tr} = T_{t\theta} = T_{\phi r} = T_{\phi\theta} = 0$. Thus, taking into account this fact, and considering that L_F is regular, we obtain the following relationships

$$g^{\theta\theta} F_{t\theta} F_{r\theta} = F_{t\phi} (g^{t\phi} F_{tr} + g^{\phi\phi} F_{\phi r})\,, \tag{83}$$

$$-g^{\theta\theta} F_{\phi\theta} F_{r\theta} = F_{t\phi} (g^{tt} F_{tr} + g^{t\phi} F_{\phi r})\,, \tag{84}$$

$$-g^{rr} F_{tr} F_{r\theta} = F_{t\phi} (g^{t\phi} F_{t\theta} + g^{\phi\phi} F_{\phi\theta})\,, \tag{85}$$

$$g^{rr} F_{\phi r} F_{r\theta} = F_{t\phi} (g^{tt} F_{t\theta} + g^{t\phi} F_{\phi\theta})\,. \tag{86}$$

Now, rewriting Eqs. (83)-(84) in order of $F_{t\theta}$ and $F_{\phi\theta}$, respectively, and introducing these in Eq. (85), we finally arrive at the following relationship

$$-g^{rr} N^2 \sin^2\theta\, F_{r\theta}^2 = F_{t\phi}^2\,, \tag{87}$$

from which we obtain that $(1 - b/r) < 0$, implying that $r < b$ for all values of r except at the throat. However, this restriction is in clear contradiction with the definition of a traversable wormhole, where the condition $(1 - b/r) > 0$ is imposed.

The same restriction can be inferred from the electromagnetic field equations $(F^{\mu\nu} L_F)_{;\mu} = 0$, which provide the following relationships

$$F^{rt} [\ln(L_F)]_{,r} + F^{\theta t} [\ln(L_F)]_{,\theta} = -F^{\mu t}{}_{;\mu}\,, \tag{88}$$

$$F^{r\phi} [\ln(L_F)]_{,r} + F^{\theta\phi} [\ln(L_F)]_{,\theta} = -F^{\mu\phi}{}_{;\mu}\,. \tag{89}$$

These can be rewritten in the following manner

$$[\ln(L_F)]_{,r} = \frac{F^{\theta t} F^{\mu\phi}{}_{;\mu} - F^{\theta\phi} F^{\mu t}{}_{;\mu}}{F^{rt} F^{\theta\phi} - F^{r\phi} F^{\theta t}}\,, \tag{90}$$

$$[\ln(L_F)]_{,\theta} = \frac{F^{r\phi} F^{\mu t}{}_{;\mu} - F^{rt} F^{\mu\phi}{}_{;\mu}}{F^{rt} F^{\theta\phi} - F^{r\phi} F^{\theta t}}\,. \tag{91}$$

Now, a crucial point to note is that to have a solution, the term in the denominator, $F^{rt}F^{\theta\phi} - F^{r\phi}F^{\theta t}$, should be non-zero, and can be expressed as

$$F^{rt}F^{\theta\phi} - F^{r\phi}F^{\theta t} = g^{rr} g^{\theta\theta} \left[g^{tt} g^{\phi\phi} - (g^{t\phi})^2\right] \times (F_{tr} F_{\phi\theta} - F_{t\theta} F_{\phi r})\,. \tag{92}$$

However, using the Eqs. (83)-(86), we may obtain an alternative relationship, given by

$$F^{rt}F^{\theta\phi} - F^{r\phi}F^{\theta t} = \left(g^{rr} g^{\theta\theta} \frac{F_{r\theta}}{F_{t\phi}}\right)^2 (F_{tr} F_{\phi\theta} - F_{t\theta} F_{\phi r})\,. \tag{93}$$

Confronting both relationships, we obtain that $(1 - b/r) < 0$, implying that $r < b$ for all values of r except at the throat, which, as before, is in clear contradiction with the definition of a traversable wormhole.

If we consider the individual cases of $F_{tr} = 0$, $F_{\phi r} = 0$, $F_{t\theta} = 0$ or $F_{\phi\theta} = 0$, separately, it is a simple matter to verify that Eqs. (83)-(86) impose the restriction $N^2 < 0$, which does not satisfy the wormhole conditions.

For the specific case of $F_{t\phi} = 0$ (with $F_{r\theta} \neq 0$), the restrictions $F_{tr} = F_{\phi r} = F_{t\theta} = F_{\phi\theta} = 0$ are imposed. Taking into account these impositions one verifies that from Eq. (82), we have $T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = 0$, which is not compatible with the geometric conditions imposed by Eq. (80).

Considering $F_{r\theta} = 0$ (with $F_{t\phi} \neq 0$), one readily verifies from Eqs. (83)-(86) the existence of an event horizon, i.e., $N = 0$, rendering the wormhole geometry non-traversable.

The specific case of $F_{t\phi} = 0$ and $F_{r\theta} = 0$ also obeys Eqs. (83)-(86), and needs to be analyzed separately. To show that this case is also in contradiction to the wormhole conditions at the throat, we shall consider the following stress-energy tensor components, for $F_{t\phi} = 0$ and $F_{r\theta} = 0$

$$T_{\hat{t}\hat{t}} = -L - \frac{(1-b/r)}{N^2} (F_{tr} + \omega F_{\phi r})^2 L_F - \frac{1}{N^2 r^2 K^2} (F_{t\theta} + \omega F_{\phi\theta})^2 L_F\,, \tag{94}$$

$$T_{\hat{r}\hat{r}} = L + \frac{(1-b/r)}{N^2} (F_{tr} + \omega F_{\phi r})^2 L_F - \frac{(1-b/r)}{r^2 K^2 \sin^2\theta} F_{\phi r}^2\, L_F\,, \tag{95}$$

$$T_{\hat{\theta}\hat{\theta}} = L + \frac{1}{N^2 r^2 K^2} (F_{t\theta} + \omega F_{\phi\theta})^2 L_F - \frac{1}{r^4 K^4 \sin^2\theta} F_{\phi\theta}^2\, L_F\,, \tag{96}$$

$$T_{\hat{\phi}\hat{\phi}} = L - \frac{(1-b/r)}{r^2 K^2 \sin^2\theta} F_{\phi r}^2\, L_F - \frac{1}{r^4 K^4 \sin^2\theta} F_{\phi\theta}^2\, L_F\,. \tag{97}$$

An important result deduced from the above components is the following

$$T_{\hat{t}\hat{t}}+T_{\hat{r}\hat{r}}+T_{\hat{\theta}\hat{\theta}}-T_{\hat{\phi}\hat{\phi}}=0 \tag{98}$$

for all points in the geometry.

Now, considering the Einstein tensor components, Eqs. (74)-(78), evaluated at the throat, along the rotation axis, and using the Einstein field equation, we verify the following relationship

$$G_{\hat{t}\hat{t}}+G_{\hat{r}\hat{r}}+G_{\hat{\theta}\hat{\theta}}-G_{\hat{\phi}\hat{\phi}}=\frac{e^{-\mu}\mu_r(rK)_r}{rK}\,, \tag{99}$$

which is always positive. We have taken into account that the functions N and K are regular, so that their θ derivatives vanish along the rotation axis, $\theta=0,\ \pi$, as emphasized above. Therefore, through the Einstein field equation, we verify that relationship (99) is not compatible with condition (98), and therefore rules out the existence of rotating wormholes for this specific case of $F_{t\phi}=0$ and $F_{r\theta}=0$.

4. CONCLUSION

In this work we have explored the possibility of the existence of $(2+1)$ and $(3+1)-$dimensional static, spherically symmetric and stationary, axisymmetric traversable wormholes coupled to nonlinear electrodynamics. For the static and spherically symmetric wormhole spacetimes, we have found the presence of an event horizon, and that the NEC is not violated at the throat, proving the non-existence of these exotic geometries within nonlinear electrodynamics. It is perhaps important to emphasize that for the $(2+1)-$dimensional case we found an extremely troublesome aspect of the geometry, as in order to construct a traversable wormhole, singularities appear in the physical fields. This particular aspect of the geometry is in clear contradiction to the model construction of nonlinear electrodynamics, founded on a principle of finiteness, that a satisfactory theory should avoid physical quantities becoming infinite [1]. Thus, imposing that the physical quantities be non-singular, we verify that the general solution corresponds to a non-traversable wormhole geometry. We also point out that the non-existence of $(3+1)-$dimensional static and spherically symmetric traversable wormholes is consistent with previous results [14].

For the $(2{+}1)-$dimensional stationary and axisymmetric wormhole, we have verified the presence of an event horizon, rendering a non-traversable wormhole geometry. Relatively to the $(3+1)-$dimensional stationary and axially symmetric wormhole geometry, we have found that the field equations impose specific conditions that are incompatible with the properties of wormholes. Thus, we have showed that for the general cases of solutions outlined above the non-existence of traversable wormholes within the context of nonlinear electrodynamics. Nevertheless, it is important to emphasize that regular magnetic time-dependent traversable wormholes do exist coupled to nonlinear electrodynamics [26].

In the analysis outlined in this paper, we have considered general relativity coupled to nonlinear electrodynamics, with the gauge-invariant electromagnetic Lagrangian $L(F)$ depending on a single invariant F given by $F\sim F^{\mu\nu}F_{\mu\nu}$. An interesting issue to pursue would be the inclusion, in addition to F, of another electromagnetic field invariant $G\sim {}^{*}F^{\mu\nu}F_{\mu\nu}$. This latter inclusion would possibly add an interesting analysis to the solutions found in this paper.

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